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TOPOLOGICAL QUANTUM FIELD THEORY:

A PROGRESS REPORT[1]

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Abstract

A brief introduction to Topological Quantum Field Theory as well as a description of recent progress made in the field is presented. I concentrate mainly on the connection between Chern-Simons gauge theory and Vassiliev invariants, and Donaldson theory and its generalizations and Seiberg-Witten invariants. Emphasis is made on the usefulness of these relations to obtain explicit expressions for topological invariants, and on the universal structure underlying both systems.

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1 Introduction

During the last decade we have witnessed a striking new relation between mathematics and physics. This relation connects many of the most advanced ideas in the two fields involved, topology and quantum physics. On the one side, the most sophisticated topological invariants of three and four-dimensional manifolds are encountered. On the other side, the most recent achievements in quantum field theory play a salient role. It is remarkable to observe that precisely these low dimensions in which topology has shown to present important features are the dimensions where many interesting quantum field theories are renormalizable.

Though connections between quantum physics and topology can be traced back to the fifties, it is in the eighties when a new and unprecedented kind of relation between the two takes place. In 1982 E. Witten [1] considered $N = 2$ supersymmetric sigma models in two dimensions and rewrote Morse theory in the language of quantum field theory. Furthermore, he constructed out of those models a refined version of Morse theory known nowadays as Morse-Witten theory. Witten's arguments in [1] made use of functional integrals and therefore can be regarded as non-rigorous. Nevertheless, some years later A. Floer reformulated Morse-Witten theory providing a rigorous mathematical structure [2]. This trend in which some mathematical structure is first constructed by quantum field theory methods and then reformulated in a rigorous mathematical ground constitutes one of the tendencies in this new relation between topology and physics.

The influence of M. Atiyah [3] on E. Witten in the fall of 1987 culminated with the construction by the latter of the first topological quantum field theory (TQFT) in January 1988 [4]. The quantum theory turned out to be a "twisted" version of $N = 2$ supersymmetric Yang-Mills. This theory, whose existence was conjectured by M. Atiyah, is related to Donaldson invariants for four-manifolds [5], and it is known nowadays as Donaldson-Witten theory.

In 1988 E. Witten formulated also two models which have been of fundamental importance in two and three dimensions: topological sigma models [6] and Chern-Simons gauge theory [7]. The first one can be understood as a twist of the $N = 2$ supersymmetric sigma model considered by Witten in his work on Morse theory [1], and is related to Gromov invariants [8]. The second one is not the result of a twist but a model whose action is the integral of the Chern-Simons form. In this case the corresponding topological invariants are knot and link invariants as the Jones polynomial [9] and its generalizations.

TQFT provided a new point of view to study the topological invariants which were discovered only a few years before the formulation of this type of quantum theory. One of the important aspects of this new approach is that they could be generalized in a variety of directions. Since 1988 there are two main lines of work: on the one hand, the rigorous constructions (without using functional integration) of the generalizations predicted by TQFT; on the other hand, the use of quantum field theory techniques to analyze and compute the generalized invariants. Both lines of work have provided a very valuable outcome but, perhaps, the most striking results have been achieved by the quantum field theory line of thought.

TQFTs have been studied from both, perturbative and non-perturbative approaches. In physical theories it is well known that both approaches provide very valuable information on the features of the models under consideration. In general, non-perturbative methods are less developed than perturbative ones. However, precisely in $N = 2$ super-

symmetric theories, the ones intimately related to TQFTs, important progress has been done recently [10]. It is also important to notice that TQFTs are in general much simpler than their physical counterparts and one expects that the use of these methods is much more tractable.

In three dimensions, non-perturbative methods have been applied to Chern-Simons gauge theory to obtain properties of knot and link invariants as well as general procedures for their computation. On the other hand, perturbative methods have provided an integral representation for Vassiliev invariants [11] which, among other things, allows to extend the formulation of these invariants to arbitrary smooth three-manifolds. Vassiliev invariants are strong candidates to classify knots and links and they have been much studied during the last years.

In four dimensions, perturbative methods show that Donaldson-Witten theory is related to Donaldson invariants. On the other hand, non-perturbative methods indicate that those invariants are related to other rather different topological invariants which are called Seiberg-Witten invariants. In sharp contrast to Donaldson invariants, which are defined on the moduli spaces of instantons, Seiberg-Witten invariants are associated to moduli spaces of abelian monopoles [12, 42]. Recently, Donaldson-Witten theory has been generalized to a theory involving non-abelian monopoles [13]. This provides a rich set of new topological invariants which opens new perspectives in the study of four-manifolds. Nevertheless, there are indications that these new invariants can also be written, at least in some situations, in terms of Seiberg-Witten invariants. Therefore, it might happen that no new topological information is gained.

The situation for three and four-dimensional TQFTs is shown in Table 1. These theories seem to share a common structure. Their topological invariants are rich and can be labeled with group theoretical data: Wilson lines for different representations and gauge groups (Jones polynomial and its generalizations) and non-abelian monopoles for different representations and gauge groups (generalized Donaldson polynomials). However, these invariants can all be written in terms of topological invariants which are independent of the group and representations chosen: Vassiliev invariants and Seiberg-Witten invariants, respectively. Both depend strictly on the topology. The group-theoretical data labeling generalized Jones and Donaldson polynomials enter in the coefficients of the expressions of these polynomials as a power series in Vassiliev and Seiberg-Witten invariants, respectively.

	$d = 3$	$d = 4$
perturbative	Vassiliev	Donaldson
non-perturbative	Jones	Seiberg-Witten

Table 1: Topological invariants in the perturbative and the non-perturbative regimes for $d = 3$ and $d = 4$.

The resemblance between the two pictures is very appealing. Nevertheless, there are important differences which rise some important questions. In the case of knot theory, Vassiliev invariants constitute an infinite set. However, in Donaldson theory, for the cases studied so far, only a finite set of invariants seems to play a relevant role. One would like to know if this is general or if this fact is just a peculiarity of the only two cases (gauge group $SU(2)$ without matter and with one multiplet of matter in the fundamental representation) which have been studied so far. The general picture of non-perturbative

$N = 2$ supersymmetric Yang-Mills theories seems to suggest that the set of invariants entering the expressions for the generalized Donaldson polynomials is going to be finite. However, one might find unexpected results by studying different kinds of matter. Much work in this direction has to be done.

In this talk I will concentrate mainly on a description of some of the recent progress made in the field. I will begin by introducing the framework of TQFTs in sec. 2 and their methods of construction in sec. 3. Excellent reviews on these aspects are available [14], [15], and I refer the reader to those for details. In sec. 4 I describe the progress made in the last years in three dimensions while in sec. 5 I do the same in four dimensions. Two-dimensional TQFT, a very active subject during the last few years, lie outside the scope of this talk. Finally, some concluding remarks are presented in sec. 6.

2 Topological Quantum Field Theory

In this section we present the most general structure of a TQFT from a functional integral point of view. As in ordinary quantum field theory, the functional integration involved is not in general well defined. Similarly to the case of ordinary quantum field theory this has led to the construction of an axiomatic approach [16]. In this section, however, we are not going to describe this approach. We will concentrate on the functional integral point of view. Although not well defined in general, this is the approach which more success has gathered.

Our basic topological space will be an n-dimensional Riemannian manifold M endowed with a metric $g_{\mu\nu}$. Let us consider on it a set of fields $\{\phi_i\}$, and let $S(\phi_i)$ be a real functional of these fields which will be regarded as the action of the theory. We will consider "operators", $\mathcal{O}_\alpha(\phi_i)$, which are in general arbitrary functionals of the fields. In TQFT these functionals are real functionals labeled by some set of indices α carrying topological or group-theoretical data. The vacuum expectation value (vev) of a product of these operators is defined as:

$$\langle \mathcal{O}_{\alpha_1} \mathcal{O}_{\alpha_2} \cdots \mathcal{O}_{\alpha_p} \rangle = \int [D\phi_i] \mathcal{O}_{\alpha_1}(\phi_i) \mathcal{O}_{\alpha_2}(\phi_i) \cdots \mathcal{O}_{\alpha_p}(\phi_i) \exp\left(-S(\phi_i)\right). \qquad (2.1)$$

A quantum field theory is considered topological if the following relation is satisfied:

$$\frac{\delta}{\delta g^{\mu\nu}} \langle \mathcal{O}_{\alpha_1} \mathcal{O}_{\alpha_2} \cdots \mathcal{O}_{\alpha_p} \rangle = 0, \qquad (2.2)$$

i.e., if the vevs of some set of selected operators is independent of the metric $g_{\mu\nu}$ on M. If such is the case those operators are called "observables".

There are two ways to guarantee, at least formally, that condition (2.2) is satisfied. The first one corresponds to the situation in which both, the action $S(\phi_i)$, as well as the operators $\mathcal{O}_{\alpha_i}$ are metric independent. These TQFTs are called of Schwarz type. The most important representative is Chern-Simons gauge theory. The second one corresponds to the case in which there exist a symmetry, whose infinitesimal form will be denoted by δ, satisfying the following properties:

$$\delta \mathcal{O}_{\alpha_i} = 0, \qquad T_{\mu\nu} = \delta G_{\mu\nu}, \qquad (2.3)$$

where $T_{\mu\nu}$ is the energy-momentum tensor of the theory, *i.e.*,

$$T_{\mu\nu}(\phi_i) = \frac{\delta}{\delta g^{\mu\nu}} S(\phi_i). \tag{2.4}$$

The fact that δ in (2.3) is a symmetry of the theory implies that the transformations $\delta\phi_i$ of the fields are such that both $\delta S(\phi_i) = 0$ and $\delta\mathcal{O}_{\alpha_i}(\phi_i) = 0$. Conditions (2.3) lead, at least formally, to the following relation for vevs:

$$\frac{\delta}{\delta g^{\mu\nu}}\langle\mathcal{O}_{\alpha_1}\mathcal{O}_{\alpha_2}\cdots\mathcal{O}_{\alpha_p}\rangle = -\int[D\phi_i]\mathcal{O}_{\alpha_1}(\phi_i)\mathcal{O}_{\alpha_2}(\phi_i)\cdots\mathcal{O}_{\alpha_p}(\phi_i)T_{\mu\nu}\exp\big(-S(\phi_i)\big)$$

$$= -\int[D\phi_i]\delta\Big(\mathcal{O}_{\alpha_1}(\phi_i)\mathcal{O}_{\alpha_2}(\phi_i)\cdots\mathcal{O}_{\alpha_p}(\phi_i)G_{\mu\nu}\exp\big(-S(\phi_i)\big)\Big) = 0, \tag{2.5}$$

which implies that the quantum field theory can be regarded as topological. This second type of TQFTs are called of Witten type. One of its main representatives is the theory related to Donaldson invariants, which, as already indicated, is a twisted version of $N = 2$ supersymmetric Yang-Mills theory. It is important to remark that the symmetry δ must be a scalar symmetry, *i.e.*, that its symmetry parameter must be a scalar. The reason is that, being a global symmetry, this parameter must be covariantly constant and for arbitrary manifolds this property, if it is satisfied at all, implies strong restrictions unless the parameter is a scalar.

3   Construction Methods

As indicated in the introduction, the first TQFT was constructed performing a twist on $N = 2$ supersymmetric Yang-Mills. Since then other construction methods have been used. These methods have been useful because, on the one hand, they have led to the construction of new TQFTs, and, on the other hand they have provided a better geometric understanding of the meaning of the TQFT under consideration. In the first group stand out the method of twisting $N = 2$ supersymmetry, as well as methods based on the "BRST" formalism also called cohomological approach. In the second group the Mathai-Quillen formalism is the most successful one. Nowadays it can be considered as the best approach although often the information learned from the other approaches turns out to be very valuable. In this section I will describe very briefly some of these approaches.

3.1 TWISTING $N = 2$ SUPERSYMMETRY

In this approach the starting point is an $N = 2$ supersymmetric quantum field theory. The basic ingredient of this method consists of extracting a scalar symmetry out of the $N = 2$ supersymmetry. Let us consider the case of $d = 4$. In $\mathbf{R}^4$ the global symmetry group when $N = 2$ supersymmetry is present is $H = SU(2)_L \otimes SU(2)_R \otimes SU(2)_I \otimes U(1)_{\mathcal{R}}$ where $\mathcal{K} = SU(2)_L \otimes SU(2)_R$ is the rotation group and $SU(2)_I \otimes U(1)_{\mathcal{R}}$ is the internal symmetry group. The supercharges Q_α^i and $\overline{Q}_{i\dot\alpha}$ which generate $N = 2$ supersymmetry have the following transformations under H:

$$Q_\alpha^i \ (\tfrac{1}{2}, 0, \tfrac{1}{2})^1, \qquad \overline{Q}_{i\dot\alpha} \ (0, \tfrac{1}{2}, \tfrac{1}{2})^{-1}, \tag{3.1}$$

where the superindex denotes the $U(1)_{\mathcal{R}}$ charge and the numbers within parentheses the representations under each of the factors in $SU(2)_L \otimes SU(2)_R \otimes SU(2)_I$.

The twist consists of considering as the rotation group the group $\mathcal{K}' = SU(2)'_L \otimes SU(2)_R$ where $SU(2)'_L$ is the diagonal subgroup of $SU(2)_L \otimes SU(2)_I$. This implies that the isospin index i becomes a spinorial index α: $Q^i_\alpha \to Q^\beta_\alpha$ and $\overline{Q}_{i\dot{\beta}} \to G_{\alpha\dot{\beta}}$. Precisely the trace of Q^β_α is chosen as the generator of the scalar symmetry: $Q = Q^\alpha_\alpha$. Under the new global group $H' = \mathcal{K}' \otimes U(1)_\mathcal{R}$, the symmetry generators transform as:

$$G_{\alpha\dot{\beta}} \ (\frac{1}{2},\frac{1}{2})^{-1}, \qquad Q_{(\alpha\beta)} \ (1,0)^1, \qquad Q \ (0,0)^1. \tag{3.2}$$

Once the scalar symmetry is found we must study if, as stated in (2.3), the energy-momentum tensor is exact, $i.e.$, if it can be written as the transformation of some quantity under Q. The $N = 2$ supersymmetry algebra gives a necessary condition for this to hold. Notice that, after the twisting, such an algebra becomes:

$$\{Q^i_\alpha, \overline{Q}_{j\dot{\beta}}\} = \delta^i_j P_{\alpha\dot{\beta}} \longrightarrow \{Q, G_{\alpha\beta}\} = P_{\alpha\dot{\beta}}, \tag{3.3}$$

where $P_{\alpha\dot{\beta}}$ is the momentum operator of the theory. Certainly (3.3) is only a necessary condition for the theory being topological. However, up to date, for all the $N = 2$ supersymmetric models whose twisting has been studied, the relation on the right hand side of (3.3) has become valid for the whole energy-momentum tensor.

In $\mathbf{R}^4$ the original and the twisted theory are equivalent. However, for arbitrary manifolds they are certainly different due to the fact that their energy-momentum tensors are different. The twisting changes the spin quantum numbers of the fields entering the theory and therefore their couplings to the metric on M are different.

As in all theories of Witten type the observables are obtained from the Q-cohomology of the fields of the theory. It is rather straightforward to prove that if one of the operators entering (2.1) is Q-exact the corresponding vacuum expectation value vanishes. In most of the cases the Q-cohomology which must be studied is an equivariant cohomology. Certainly this study must be done among the operators which are invariant under Q^2. For example, for Yang-Mills theory, the $N = 2$ supersymmetry algebra closes up to gauge transformations (Wess-Zumino gauge). This implies that Q^2 is a gauge transformation and therefore the Q cohomology must be studied on gauge invariant operators.

3.2 BRST approach

Soon after Donaldson-Witten theory was formulated [4], the theory was reobtained by another method based on the application of BRST gauge-fixing techniques [17]. This approach automatically generates a scalar symmetry and an energy-momentum tensor which is Q-exact. The basic ingredients of this approach are a set of basic fields, ϕ_i, and a set of basic equations $s^{(\alpha)}(\phi_i) = 0$. The main idea is to assume that the theory possesses a symmetry which corresponds to arbitrary variations of the basic fields and then BRST gauge-fix this symmetry using as gauge fixing function the basic equations $s^{(\alpha)}(\phi_i) = 0$. As classical action one takes zero and therefore the construction automatically leads to an action which is Q-exact. The Q-exactness of the energy-momentum follows if the Q-transformations commute with the variations respect to the metric $g_{\mu\nu}$. In the gauge-fixing process new fields are generated: ghost, antighost and auxiliary fields. As in the previous case the observables are obtained studying the corresponding equivariant Q-cohomology.

3.3 Mathai-Quillen formalism

This formalism is the most geometrical one among all the approaches leading to the construction of TQFT. It can be applied to any Witten type theory. Based on the work

by Mathai and Quillen [18] it was first implemented in the framework of TQFT by Atiyah and Jeffrey [19]. The basic idea behind this formalism is the extension to the infinite-dimensional case of ordinary finite-dimensional geometrical constructions. Soon after the formulation of the first TQFTs it became clear that the partition function associated to most of these theories corresponds to the Euler class of certain vector bundle related to the space of solutions of the basic equations of the theory (moduli space). In the finite-dimensional case there are many forms to obtain the Euler class. The Mathai-Quillen formalism basically consists of the generalization of one of these forms to the infinite-dimensional case.

Let us consider a vector bundle E of dimension $2m$ on a manifold M. This bundle has its Euler class in $H^{2m}(M)$. If $\dim M = 2m$ and M is compact and orientable, the integration of one representative of the Euler class, for example for the tangent bundle to M the one supplied by the Gauss-Bonet theorem, ω, leads to the Euler number,

$$\varepsilon(E) = \int_M \omega. \tag{3.4}$$

The equations that one obtains for the partition function of a TQFT have a similar structure. However, such a formula is hard to implement in this context because it lacks information on the main ingredient of a TQFT of Witten type: the moduli space of solutions of its basic equations. Fortunately, a most general form of (3.4) based on the Thom class is available. Furthermore, this formula, besides a dependence on a connection on the vector bundle E, it also depends on a section which certainly opens a room to introduce the basic equations. This formula has the form,

$$\varepsilon(E) = \int_M \omega_s = \int_M s^* U, \tag{3.5}$$

where U is a representative of the Thom class. The generalization of this expression to the infinite-dimensional case leads to the Mathai-Quillen formalism. We will not describe here the details of this formalism. We refer the reader to [15, 20, 21, 22] where excellent reviews of this approach are presented.

3.4 SCHWARZ TYPE THEORIES

The previous methods considered refer to the case of Witten type theories. In the case of Schwarz type theories one must first construct an action which is independent of the metric $g_{\mu\nu}$. The method is best illustrated by considering an example. Let us take into consideration the most interesting of this type of theories: Chern-Simons gauge theory [7]. This is a three-dimensional theory whose action is the integral of the Chern-Simons form associated to a gauge connection A corresponding to a group G:

$$S_{\mathrm{CS}}(A) = \int_M \mathrm{Tr}(A \wedge dA + \frac{2}{3} A \wedge A \wedge A). \tag{3.6}$$

Observables must be constructed out of operators which do not contain the metric $g_{\mu\nu}$. In gauge invariant theories, as it is the case, one must also demand invariance under gauge transformations for these operators. An important set of the observables in Chern-Simons gauge theory is constituted by the trace of the holonomy of the gauge connection A in some representation R along a 1-cycle γ, *i.e.*, the Wilson line:

$$\mathrm{Tr}_R(\mathrm{Hol}_\gamma(A)) = \mathrm{Tr}_R \mathrm{P} \exp \int_\gamma A. \tag{3.7}$$

4   Chern-Simons gauge theory

We have already introduced Chern-Simons gauge theory in the previous subsection. This theory has a tremendous importance because the vevs of its observables are related to knot and link invariants of polynomial type as the Jones polynomial and its generalizations (HOMFLY, Kauffman, Akutsu-Wadati, etc.).

The data in Chern-Simons gauge theory are the following: a differentiable three-manifold M which I will take to be compact, a gauge group G, which will be taken simple and compact, and an integer parameter k. Once these are chosen the observables are labeled by representations R_i and embeddings γ_i of S^1 into M. They lead to the following vevs:

$$\langle \text{Tr}_{R_1} \text{Pe}^{\int_{\gamma_1} A} \ldots \text{Tr}_{R_n} \text{Pe}^{\int_{\gamma_n} A} \rangle = \int [DA] \text{Tr}_{R_1} \text{Pe}^{\int_{\gamma_1} A} \ldots \text{Tr}_{R_n} \text{Pe}^{\int_{\gamma_n} A} \text{e}^{\frac{ik}{4\pi} S_{\text{CS}}(A)}. \qquad (4.1)$$

The reason for k being integer-valued is gauge invariance. Under a gauge transformation $A \to g^{-1} dg + g^{-1} A g$ where $g : M \to G$ the Wilson lines are invariant but the action of the theory transform as:

$$\frac{ik}{4\pi} S_{\text{CS}}(A) \longrightarrow \frac{ik}{4\pi} S_{\text{CS}}(A) - \frac{ik}{12\pi} \int_M \text{Tr}(g^{-1} dg \wedge g^{-1} dg \wedge g^{-1} dg). \qquad (4.2)$$

The second term in this equation has the form $-4i\pi y k w(g)$ where $w(g)$ is an integer, the winding number of the map g, and y is the Dynkin index of the fundamental representation of G which is half-integer. If k is an integer, the exponential of $\frac{ik}{4\pi} S_{\text{CS}}(A)$ is gauge invariant.

The non-perturbative analysis of the theory shows that the invariants associated to the observables (4.1) are knot and link invariants with the same properties as the Jones polynomial and its generalizations. If one considers $M = S^3$, $G = SU(2)$, and takes all the Wilson lines entering (4.1) in the fundamental representation, the non-perturbative analysis proves that the vevs associated to three links whose only difference is in an overcrossing, in an undercrossing or in no-crossing, satisfy the following relation:

where $q = \exp(2\pi i/(2yk + g^\vee))$ being $g^\vee$ the dual Coxeter number of the group G. These are precisely the skein rules which define the Jones polynomial. The great advantage of Chern-Simons gauge theory is that it allows to generalize very simply these invariants to other groups and other representations. The HOMFLY [24] and the Kauffman [25] polynomials are obtained after considering the fundamental representation of the groups $SU(N)$ and $SO(N)$, respectively. The Akutsu-Wadati [26] or colored Jones polynomial is obtained considering the group $SU(2)$ with Wilson lines in different representations. Other non-perturbative methods have allowed to obtain these invariants for classes of knots and links as, for example, torus knots and links [27]. Methods for general computations of these invariants have been proposed in [28] and [29].

From the point of view of perturbation theory, Chern-Simons gauge theory has led to important results. Furthermore, this theory has been studied in both, the Hamiltonian (non-covariant) and the Lagrangian (covariant) approaches providing a variety of interesting results. Besides, in each of these cases it is possible to analyze the theory in different gauges obtaining in this way remarkable relations.

The perturbative formulation of a quantum field theory is contained in its Feynman rules. In the case of Chern-Simons gauge theory these Feynman rules lead to the following general power series expansion for the vev of a Wilson line [30]:

$$\frac{\langle \text{Tr}_R \text{P} \exp \int_\gamma A \rangle}{\langle 1 \rangle} = d(R) \sum_{i=0}^{\infty} \sum_{j=1}^{d_i} \alpha_{ij}(\gamma) r_{ij}(R) x^i, \tag{4.3}$$

where $r_{ij}(R)$ (group factors) contain all the dependence on the representation and the group which has been chosen, and $\alpha_{ij}(\gamma)$ (geometric factors) all the dependence on the path γ and the three-dimensional manifold where this path is embedded. In (4.3) i corresponds to the order in perturbation theory and $x = i\pi/yk$. The integers d_i denote the number of independent group factors. Previous analysis of the lowest order of the series (4.3) can be found in [31] and [32].

The first problem that one has to face to study the expansion (4.3) is the classification of the independent group factors. The Feynman rules of the theory are such that this problem can be stated in terms of trivalent graphs. Let Γ be the set of all diagrams containing a circle and a trivalent graph. The set of independent group factors is isomorphic to Γ modulo the AS, IHX and STU relations given in [33]. Let us denote by $\mathcal{B}$ the resulting set of diagrams. This set has a natural grading (1/2 the number of trivalent vertices) which precisely coincides with the order in perturbation theory (the power of x in (4.3)). The number of elements of $\mathcal{B}$ at order i are therefore the quantities d_i appearing in (4.3). These numbers are known only up to order $i = 9$ [33] and are given in Table 2.

i	1	2	3	4	5	6	7	8	9
d_i	0	1	1	3	4	9	14	27	44

Table 2: Number of independent group factors

In the power series expansion (4.3) the group factor is independent of the knot which has been chosen. All the dependence on the knot is in the geometric factors. Since the total sum is a topological invariant and at each order the geometric factors multiply quantities which are independent, the geometric factors are knot invariants. Chern-Simons gauge theory supplies an infinite set of knot invariants with a natural grading given by the order in perturbation theory. Furthermore, it provides an explicit integral representation for each of them. At order $i = 2$, for the case in which $M = \mathbf{R}^3$, one has [31]:

$$\begin{aligned}
\alpha_{21}(\gamma) &= \frac{1}{4} \oint dx^\mu \int^x dy^\nu \int^y dz^\rho \int^z dw^\sigma \Delta_{\mu\rho}(x-z)\Delta_{\nu\sigma}(y-w) \\
&- \frac{1}{16} \oint dx^\mu \int^x dy^\nu \int^y dz^\rho \int_{\mathbf{R}^3} d^3\omega v_{\mu\nu\rho}(x,y,z;\omega),
\end{aligned} \tag{4.4}$$

where,

$$\Delta_{\mu\nu}(x-y) = \frac{1}{\pi}\epsilon_{\mu\rho\nu}\frac{(x-y)^\rho}{|x-y|^3}, \quad v_{\mu\nu\rho}(x,y,z;\omega) = \Delta_{\mu\sigma_1}(x-\omega)\Delta_{\nu\sigma_2}(y-\omega)\Delta_{\rho\sigma_3}(z-\omega)\epsilon^{\sigma_1\sigma_2\sigma_3}. \tag{4.5}$$

The knot invariants α_{ij} turn out to be Vassiliev invariants or of finite type. A knot invariant is of type n if after defining invariants for singular knots via the relation,

one finds that it vanishes for all knots containing n singularities. Birman and Lin showed [35] that for any polynomial invariant arising from Chern-Simons gauge theory with semisimple gauge group, after expressing q as,

$$q = \exp\left(\frac{2\pi i}{2yk + g^\vee}\right) = \exp(x), \tag{4.6}$$

and expanding in power series of x, the coefficient of x^n is an invariant of type n. This implies that the invariants α_{ij} are Vassiliev invariants of type n. Using the results by Bar-Natan in [33] based on the integral representation for Vassiliev invariants provided by Kontsevich [34] one can show that the quantities α_{ij} supplied by Chern-Simons gauge theory constitute a complete set of Vassiliev invariants.

As we have been discussing in this section, Chern-Simons gauge theory provides a covariant integral representation for Vassiliev invariants [30]. This representation has been considered recently by Bott and Taubes [36] from a different perspective. Kontsevich's representation for Vassiliev invariants [34] is not covariant. It can be obtained from Chern-Simons gauge theory by analyzing the theory in the Hamiltonian formalism [37]. Recent work shows that it can also be obtained from a Lagrangian analysis of the theory in a non-covariant gauge [38].

Using the perturbative expansion (4.3) Vassiliev invariants have been computed up to order six for all prime knots up to six crossings [30] and for arbitrary torus knots [39]. These results have been obtained using the fact that Chern-Simons gauge theory, as any of the TQFTs studied up to date, have a perturbative series expansion which is exact, *i.e.*, is equivalent to its non-perturbative counterpart.

5   Donaldson-Witten theory and its generalizations

As explained in the introduction, the twisted version of $N = 2$ supersymmetric Yang-Mills in four dimensions provides a TQFT [4] whose observables are related to Donaldson invariants. This theory has been extended recently [13] to a wide range of moduli spaces whose associated invariants turn out to satisfy unexpected relations.

Let M be a compact oriented four-dimensional manifold endowed with a metric $g_{\mu\nu}$, and let us consider on it a principal fibre bundle P with group G which will be assumed to be simple, compact and connected. Let E be the vector bundle associated to P via the adjoint representation, and let $\mathcal{A}$ be the space of G-connections on E. A connection in $\mathcal{A}$ will be denoted by A and its corresponding covariant derivative and self-dual part of its curvature by D_μ and F^+, respectively. Let us introduce the following set of fields:

$$\chi_{\mu\nu}, \ G_{\mu\nu} \ \in \ \Omega^{2,+}(M, \mathbf{g}), \qquad \psi_\mu \ \in \ \Omega^1(M, \mathbf{g}), \qquad \eta, \ \lambda, \ \phi \ \in \ \Omega^0(M, \mathbf{g}). \tag{5.1}$$

In (5.1) $\mathbf{g}$ denotes the Lie algebra associated to G. The action of the theory has the form:

$$S = \int_M d^4x \ \sqrt{g} \quad \mathrm{Tr}\Big(F^{+2} - G^2 - i\chi^{\mu\nu}D_\mu\psi_\nu + i\eta D_\mu\psi^\mu + \frac{1}{4}\phi\{\chi_{\mu\nu}, \chi^{\mu\nu}\}$$

$$+ \frac{i}{4}\lambda\{\psi_\mu, \psi^\mu\} - \lambda D_\mu D^\mu\phi + \frac{i}{2}\phi\{\eta, \eta\} + \frac{1}{8}[\lambda, \phi]^2\Big) \tag{5.2}$$

The scalar symmetry which characterizes the theory has the form:

$$
\begin{aligned}
\delta A_\mu &= \psi_\mu, & \delta \chi_{\mu\nu} &= G_{\mu\nu}, \\
\delta \psi &= d_A \phi, & \delta G_{\mu\nu} &= i[\chi_{\mu\nu}, \phi], \\
\delta \phi &= 0, & \delta \lambda &= \eta, & \delta \eta = i[\lambda, \phi].
\end{aligned}
\tag{5.3}
$$

These transformations close up to a gauge transformation and the action (5.2) is δ-exact. This implies, on the one hand, that the action is invariant under (5.3); on the other hand that its energy-momentum tensor is δ-exact and therefore that the theory is topological. The δ-cohomology associated to (5.3) was studied in [4]. For the case $G = SU(2)$ the resulting observables are based on the following operators:

$$
\begin{aligned}
W_0 &= \tfrac{1}{2}\mathrm{Tr}(\phi^2) & W_1 &= \mathrm{Tr}(\phi \wedge \psi), \\
W_2 &= \mathrm{Tr}(\tfrac{1}{2}\psi \wedge \psi + i\phi \wedge F), & W_3 &= i\mathrm{Tr}(\psi \wedge F).
\end{aligned}
\tag{5.4}
$$

These operators satisfy the descent equations, $\delta W_i = dW_{i-1}$, which allow to define the following observables:

$$
\mathcal{O}^{(k)} = \int_{\gamma_k} W_k,
\tag{5.5}
$$

where $\gamma_k \in H_k(M)$. The descent equations imply that they are δ-invariant.

The functional integral corresponding to the topological invariants of the theory has the form:

$$
\langle \mathcal{O}^{(k_1)} \mathcal{O}^{(k_2)} \cdots \mathcal{O}^{(k_p)} \rangle = \int \mathcal{O}^{(k_1)} \mathcal{O}^{(k_2)} \cdots \mathcal{O}^{(k_p)} \exp(-S/g),
\tag{5.6}
$$

where the integration has to be understood on the space of field configurations modulo gauge transformations and g is a coupling constant. Standard arguments [4] show that due to the δ-exactness of the theory the quantities obtained in (5.6) are independent of g. This implies that the observables of the theory can be obtained either in the limit $g \to 0$, where perturbative methods apply, or in the limit $g \to \infty$ where one is forced to use non-perturbative ones. The crucial point is to observe that the δ-exactness of the action implies, at least formally, that in either case the quantities obtained must be the same.

The previous argument for $g \to 0$ implies that the semiclassical approximation of the theory is exact. In this limit the contributions to the functional integral are dominated by the field configurations which minimize S. Let us assume that in the situation under consideration there are only irreducible connections. In this case the contributions from the even part of the action are given by the solutions of the equation $F^+ = 0$, *i.e.*, by instanton configurations. Being the connection irreducible there are no non-trivial solution to the classical equations for the fields λ and ϕ.

From the odd part of the action one finds that the only contributions come from the solutions to the equations,

$$
(D_\mu \psi_\nu)^+ = 0, \qquad D_\mu \psi^\mu = 0,
\tag{5.7}
$$

which are precisely the ones that define the tangent space to the space of instanton configurations. The number of independent solutions of these equations determine the dimension of the instanton moduli space $\mathcal{M}$. These dimensions can be computed with the help of index theorems. For $SU(2)$ one has: $d_{\mathcal{M}} = 8k - 3(\chi + \sigma)/2$, where χ is the Euler number and σ the signature of the manifold M, and k is the instanton number.

The fundamental contribution to the functional integral (5.6) is given by the elements of $\mathcal{M}$ and by the zero-modes of the solutions to (5.7). Once these have been obtained they must be introduced in the action and an expansion up to quadratic terms in non-zero modes must be performed. These are the only relevant terms in the limit $g \to 0$. The resulting gaussian integrations then must be performed. Due to the presence of the δ symmetry these come in quotiens whose value is ± 1. The functional integral (5.6) takes the form[3]:

$$\langle \mathcal{O}^{(k_1)} \mathcal{O}^{(k_2)} \cdots \mathcal{O}^{(k_p)} \rangle = \int_{\mathcal{M}} da_1 \cdots da_{d_{\mathcal{M}}} d\psi_1 \cdots d\psi_{d_{\mathcal{M}}} \mathcal{O}^{(k_1)} \mathcal{O}^{(k_2)} \cdots \mathcal{O}^{(k_p)} (-1)^{f(a_1, \ldots, a_{d_{\mathcal{M}}})},$$
(5.8)

where $f(a_1, \ldots, a_{d_{\mathcal{M}}}) = 0, 1$. The integration over the odd modes leads to a selection rule for the product of observables. This selection rule is better expressed making use of the quantum numbers of the fields associated to the $U(1)_{\mathcal{R}}$ symmetry inherited from $N = 2$ supersymmetry. These quantum numbers are usually called ghost numbers due to its origin in the BRST approach to TQFT. For the operators in (5.5) one has: $U(\mathcal{O}^{(k)}) = 4 - k$, and the selection rule can be written as $d_{\mathcal{M}} = \sum_{i=1}^{p} U(\mathcal{O}^{(k_i)})$.

In the case in which $d_{\mathcal{M}} = 0$, the only observable is the partition function which takes the form $\langle 1 \rangle = \sum_i (-1)^{f_i}$, where the sum is over isolated instantons. In general, the integration of the zero-modes in (5.8) leads to an antisymmetrization in such a way that one ends with the integration of a $d_{\mathcal{M}}$-form on $\mathcal{M}$. The resulting real number is a topological invariant. Notice that in the process a map $H_k(M) \longrightarrow H^k(\mathcal{M})$ has been constructed. The vevs of the theory provide polynomials in $H_{k_1}(M) \times H_{k_2}(M) \times \cdots \times H_{k_p}(M)$ which are precisely the Donaldson polynomials.

The study of Donaldson-Witten theory from a perturbative point of view has allowed to show that the vevs of the observables of this theory are related to Donaldson invariants. However, it does not provide a new method to compute them since the integral functional leads to an integration over the moduli space of instantons, which is precisely the step where the hardest problems to compute these invariants appear. From a quantum field theory we have still the possibility of studying the form of these observables from a non-perturbative point of view, *i.e.*, their study in the limit $g \to \infty$. This line of research seemed difficult to implement until very recently. However, in 1994, after the work by Seiberg and Witten [10], important progress was made in the knowledge of the non-perturbative structure of $N = 2$ supersymmetric Yang-Mills theories. These results have been immediately applied to the twisted theory leading to explicit expressions for the topological invariants in a variety of situations [12]. But perhaps the most important result achieved in this context is the existence of a relation between the moduli space of instantons and other moduli spaces such as the moduli space of abelian monopoles.

$N = 2$ supersymmetric Yang-Mills theory is asymptotically free. This means that the effective coupling constant becomes small at large energies. The perturbative methods which have been used are therefore valid at these energies. At low energies, however, those methods are not valid and one must use non-perturbative methods. From the work in [10] follows that at low energies the theory behaves as an abelian gauge theory. The effective theory is parametrized by a complex variable u which labels the vacuum structure of the theory. The most important feature of the effective theory is that there are points in the u-complex plane where monopoles and dyons become massless. These points are singular

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[3]This equation is a simplified form of the true one. A more detailed analysis shows that the field ϕ has to be replaced by its vev.

points and there are two for $SU(2)$, $u = \pm\Lambda^2$, where Λ is the dynamically generated scale of the theory. For the point $u = \Lambda^2$ the theory consists of an $N = 2$ supersymmetric abelian gauge theory coupled to a massless monopole, while for $u = -\Lambda^2$ it is coupled to a dyon. It is important to remark that the $\mathcal{R}$ symmetry of the initial theory is broken to a $\mathbf{Z}_2$ symmetry. From the breaking process one can extract the transformation which relates the behavior of the theory around one singularity in terms of its behavior around the other. This is very important because it is not possible to describe both with a local theory.

The theory around a massless monopole is an $N = 2$ supersymmetric abelian gauge theory coupled to a massless $N = 2$ hypermultiplet. This theory has its associated twisted version which is the one that must correspond to the low energy behavior of Donaldson-Witten theory or large coupling limit, $g \rightarrow \infty$. This version has been constructed in [40] using the Mathai-Quillen formalism. The structure of this theory is similar to the one of Donaldson-Witten theory. The resulting action is δ-exact and therefore one can study the theory in the weak coupling limit, which, being abelian, corresponds to the low energy limit. The main contribution to the functional integral from the even part of the action is given by the solutions to the equations,

$$F_{\mu\nu}^+ + \frac{i}{2}\overline{M}\gamma_{\mu\nu}^+ M = 0, \qquad \gamma^\mu D_\mu M = 0, \tag{5.9}$$

where M is a Weyl spinor of $U(1)$ charge one. Let us denote by L the complex line bundle to which the abelian connection A is associated. The Weyl spinor M is a section of $S^+ \otimes L$, being S^+ the positive chirality spin bundle. Of course, we are assuming that our manifold is a spin manifold . If this were not the case a similar analysis can be carried out introducing a $Spin_c$ structure. The equations (5.9) are known as monopole equations [12].

We have studied the twisted theory around one of the singular points. However, vevs must be computed integrating over the possible values of the parameter u. The crucial point which implies calculability for the twisted theory is that it is possible to argue that in a wide variety of situations the only contributions come from the singular points. For a generic value of u the possible contributions could occur if there are abelian instantons (for a generic u there are no other massless modes). Manifolds with $b_2^+ > 1$ do not possess abelian instantons and therefore for this case one just has to sum over singular points.

Let us describe first the contribution from $u = \Lambda^2$. The selection rule corresponding to the TQFT of the twisted effective theory is similar to the one for the partition function of Donaldson-Witten theory. The dimension of the moduli space of monopoles $\tilde{\mathcal{M}}$ is $d_{\tilde{\mathcal{M}}} = -(2\chi + 3\sigma)/4 + x^2/4$ where $x = -2c_1(L)$. The selection rule $d_{\tilde{\mathcal{M}}} = 0$ implies $x^2 = 2\chi + 3\sigma$. Let us denote by n_x the resulting partition function. The classes x such that $n_x \neq 0$ are called basic classes and the quantities n_x, that being the partition function of a TQFT are topological invariants, are called Seiberg-Witten invariants. Reviews on these invariants have appeared recently [22, 23].

Instead of considering products of observables as before, let us consider their generating function. We will restrict ourselves to the case in which the four-manifold M is simply connected so that $H_1(M) = H_3(M) = 0$. In this situation the only observables are the ones associated to $\Sigma_a \in H_2(M)$ and to points $x \in M$. If we denote by $I(\Sigma_a)$ the operator

$\int_{\Sigma_a} W^{(2)}$ and by $\mathcal{O}$ the one to $W^{(0)}(x)$ we must study the following vev:

$$\langle \exp\left(\sum_a \alpha_a I(\Sigma_a) + \mu\mathcal{O}\right)\rangle, \tag{5.10}$$

where α_a and μ are parameters and a runs over a basis in $H_2(M)$. At the singular point corresponding to the massless magnetic monopole the vev of this observable takes the form:

$$\exp\left(\gamma v^2 + \mu\langle\mathcal{O}\rangle\right)\sum_x n_x e^{\langle V\rangle v\cdot x}, \tag{5.11}$$

where $v^2 = \sum_{a,b}\alpha_a\alpha_b\#(\Sigma_a \cap \Sigma_b)$, $v \cdot x = \sum_a \alpha_a \int_{\Sigma_a} x$, and γ, $\langle\mathcal{O}\rangle$ and $\langle V\rangle$ are universal constants, independent of the manifold M. To motivate the origin of (5.11) we just will indicate that v^2 is generic [41], and that in the operator $I(\Sigma_a) = \int_{\Sigma_a}(i\phi F + \psi \wedge \psi/2)$ only the first term contributes. It has the form $v \cdot x$ since at the singular point ϕ takes a constant value.

Once the contribution from the first singular point has been computed, the one from the other is easily obtained using the $\mathbf{Z}_2$ symmetry. In this case this symmetry is:

$$\mathcal{O} \longrightarrow -\mathcal{O}, \qquad I(\Sigma_a) \longrightarrow -iI(\Sigma_a). \tag{5.12}$$

There is in addition a factor due to the global anomaly of the $\mathbf{Z}_2$ symmetry. It has the form i^Δ where $\Delta = (\chi + \sigma)/4$. Finally the vev (5.10) takes the form:

$$c\left(\exp\left(\gamma v^2 + \mu\langle\mathcal{O}\rangle\right)\sum_x n_x e^{\langle V\rangle v\cdot x} + i^\Delta \exp\left(-\gamma v^2 - \mu\langle\mathcal{O}\rangle\right)\sum_x n_x e^{-i\langle V\rangle v\cdot x}\right). \tag{5.13}$$

The constants c, γ, $\langle\mathcal{O}\rangle$ and $\langle V\rangle$ can be calculated comparing to the expression obtained by Kronheimer and Mrowka [46]:

$$c = 2^{1+(7\chi+11\sigma)/4}, \quad \gamma = 1, \quad \langle\mathcal{O}\rangle = 2, \quad \langle V\rangle = 1. \tag{5.14}$$

Similar methods can be used for other groups different than $SU(2)$. In general, the number of singularities is bigger. For example, for $SU(N)$ one has N singularities. Recently, models which generalize Donaldson-Witten theory have been proposed. These theories are based on non-abelian monopoles. They can be understood as a twisted version of $N = 2$ supersymmetric Yang-Mills coupled to $N = 2$ supersymmetric matter [42] (see also [43, 44, 45]). They have been proposed using the Mathai-Quillen formalism in [13]. The fact that they can be thought as twisted versions of $N = 2$ supersymmetric theories allows to use the recent results on the non-perturbative behavior of these theories. The basic equations for $SU(N)$ non-abelian monopoles are:

$$F_{\mu\nu}^{+ij} + \frac{1}{2}\left(\overline{M}_{(\mu}^j M_{\nu)}^i - \frac{\delta^{ij}}{N}\overline{M}_{(\mu}^k M_{\nu)}^k\right) = 0, \qquad \gamma^\mu D_\mu M = 0, \tag{5.15}$$

where $M^i \in \Gamma(M, S^+ \otimes E)$, i.e., M^i is a Weyl spinor in the fundamental representation of $SU(N)$. As before in the weak coupling limit there appear a structure similar to the case of Donaldson-Witten theory. For strong coupling one finds again an abelian theory [10] but this time there are three singularities related by a $\mathbf{Z}_3$ symmetry. The most important aspect of this analysis is that the vevs of observables can be expressed in terms of the Seiberg-Witten invariants n_x as in the previous case [47]. A new study of this model has appeared recently [48].

6 Concluding Remarks

The generalization of Donaldson-Witten theory opens the study of an enormous set of TQFTs. Each of these new theories can be labeled by a gauge group and a representation (or several representations if there is more than one flavor). One would like to know for which of these theories the vevs of observables can be expressed in terms of Seiberg-Witten invariants. Is this true for all of them, or Seiberg-Witten invariants are just the first of a family of invariants which allow to express all the vevs in terms of them?

A look at (5.13) reveals that almost all the topological information of the vevs is contained in the Seiberg-Witten invariants n_x. The rest are factors fixed by the group and representation (or representations) chosen. This structure is similar to the one found for Chern-Simons gauge theory in three dimensions (see (4.3)). There the group factors do not contain information on the topology of the three manifold, all the topological information is contained in the geometric factors or Vassiliev invariants. In this sense there is a parallelism between Vassiliev and Seiberg-Witten invariants: both contain the topological information of the manifold M and are universal respect to the group and representation (or representations) chosen.

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